OCR
RECOGNISING ACHIEVEMENT

Thursday 14 June 2012 – Morning

A2 GCE MATHEMATICS

4726 Further Pure Mathematics 2

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4726
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **8** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

1 Express $\operatorname{sech} 2x$ in terms of exponentials and hence, by using the substitution $u = e^{2x}$, find $\int \operatorname{sech} 2x \, dx$. **[5]**

2 A curve has polar equation $r = \cos\theta \sin 2\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Find

(i) the equations of the tangents at the pole, **[2]**

(ii) the maximum value of r, **[4]**

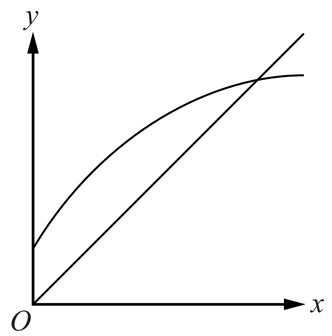
(iii) a cartesian equation of the curve, in a form not involving fractions. **[3]**

3 **(i)** By quoting results given in the List of Formulae (MF1), prove that $\tanh 2x \equiv \dfrac{2\tanh x}{1 + \tanh^2 x}$. **[2]**

(ii) Solve the equation $5\tanh 2x = 1 + 6\tanh x$, giving your answers in logarithmic form. **[6]**

4 It is given that the equation $x^4 - 2x - 1 = 0$ has only one positive root, α, and $1.3 < \alpha < 1.5$.

(i)



The diagram shows a sketch of $y = x$ and $y = \sqrt[4]{2x+1}$ for $x \geqslant 0$. Use the iteration $x_{n+1} = \sqrt[4]{2x_n + 1}$ with $x_1 = 1.35$ to find x_2 and x_3, correct to 4 decimal places. On the copy of the diagram show how the iteration converges to α. **[3]**

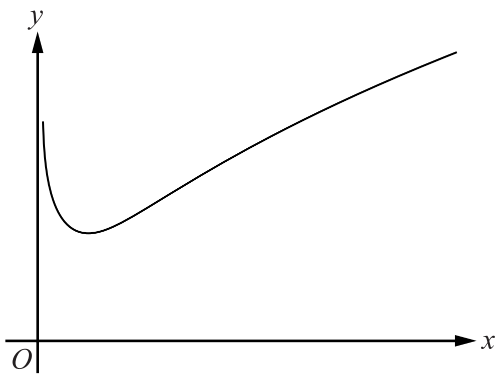
(ii) For the same equation, the iteration $x_{n+1} = \frac{1}{2}\left(x_n^4 - 1\right)$ with $x_1 = 1.35$ gives $x_2 = 1.1608$ and $x_3 = 0.4077$, correct to 4 decimal places. Draw a sketch of $y = x$ and $y = \frac{1}{2}\left(x^4 - 1\right)$ for $x \geqslant 0$, and show how this iteration does not converge to α. **[2]**

(iii) Find the positive root of the equation $x^4 - 2x - 1 = 0$ by using the Newton-Raphson method with $x_1 = 1.35$, giving the root correct to 4 decimal places. **[4]**

5 A function is defined by $f(x) = \sinh^{-1} x + \sinh^{-1}\left(\frac{1}{x}\right)$, for $x \neq 0$.

(i) When $x > 0$, show that the value of $f(x)$ for which $f'(x) = 0$ is $2\ln\left(1 + \sqrt{2}\right)$. **[5]**

(ii)



The diagram shows the graph of $y = f(x)$ for $x > 0$. Sketch the graph of $y = f(x)$ for $x < 0$ and state the range of values that $f(x)$ can take for $x \neq 0$. **[3]**

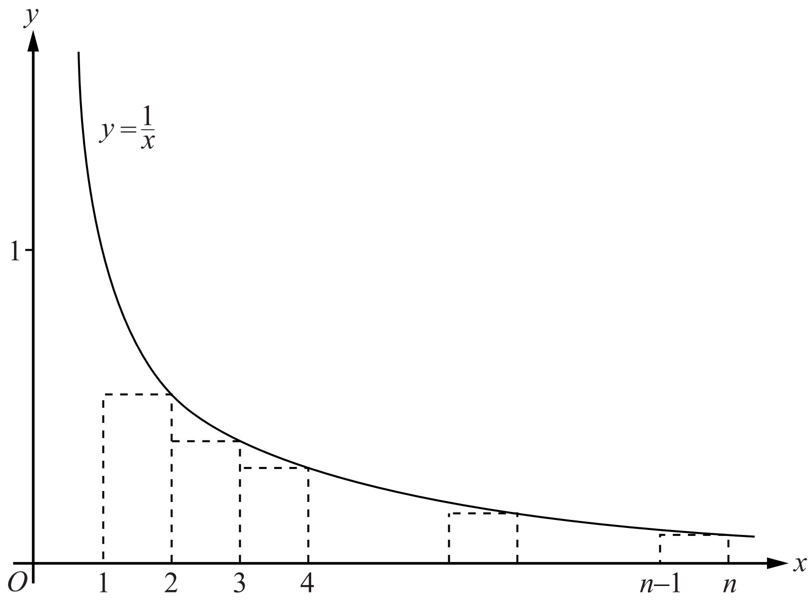
6 It is given that, for non-negative integers n,

$$I_n = \int_0^{\pi} x^n \sin x \, dx.$$

(i) Prove that, for $n \geqslant 2$, $I_n = \pi^n - n(n-1)I_{n-2}$. **[5]**

(ii) Find I_5 in terms of π. **[4]**

7



The diagram shows the curve $y = \frac{1}{x}$ for $x > 0$ and a set of $(n - 1)$ rectangles of unit width below the curve. These rectangles can be used to obtain an inequality of the form

$$\frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \ldots + \frac{1}{b} < \int_1^n \frac{1}{x}\,\mathrm{d}x.$$

Another set of rectangles can be used similarly to obtain

$$\int_1^n \frac{1}{x}\,\mathrm{d}x < \frac{1}{c} + \frac{1}{c+1} + \frac{1}{c+2} + \ldots + \frac{1}{d}.$$

(i) Write down the values of the constants a and c, and express b and d in terms of n. **[3]**

The function f is defined by $\mathrm{f}(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$, for positive integers n.

(ii) Use your answers to part **(i)** to obtain upper and lower bounds for $\mathrm{f}(n)$. **[4]**

(iii) By using the first 2 terms of the Maclaurin series for $\ln(1 + x)$ show that, for large n,

$$\mathrm{f}(n+1) - \mathrm{f}(n) \approx -\frac{n-1}{2n^2(n+1)}.$$ **[5]**

8 The curve C_1 has equation $y = \dfrac{\mathrm{p}(x)}{\mathrm{q}(x)}$, where $\mathrm{p}(x)$ and $\mathrm{q}(x)$ are polynomials of degree 2 and 1 respectively. The asymptotes of the curve are $x = -2$ and $y = \frac{1}{2}x + 1$, and the curve passes through the point $\left(-1, \frac{17}{2}\right)$.

(i) Express the equation of C_1 in the form $y = \dfrac{\mathrm{p}(x)}{\mathrm{q}(x)}$. **[4]**

(ii) For the curve C_1, find the range of values that y can take. **[4]**

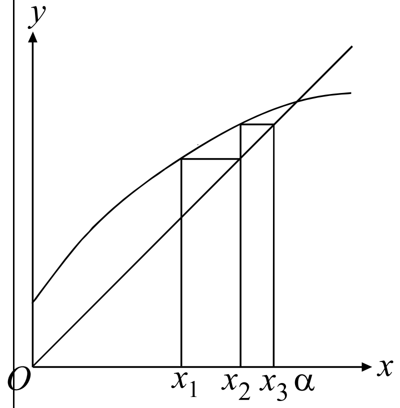
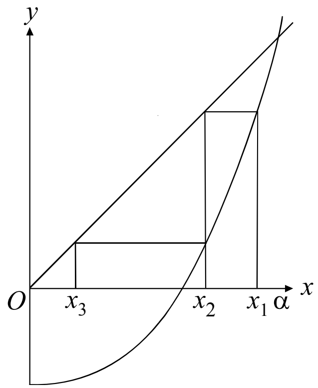
Another curve, C_2, has equation $y^2 = \dfrac{\mathrm{p}(x)}{\mathrm{q}(x)}$, where $\mathrm{p}(x)$ and $\mathrm{q}(x)$ are the polynomials found in part **(i)**.

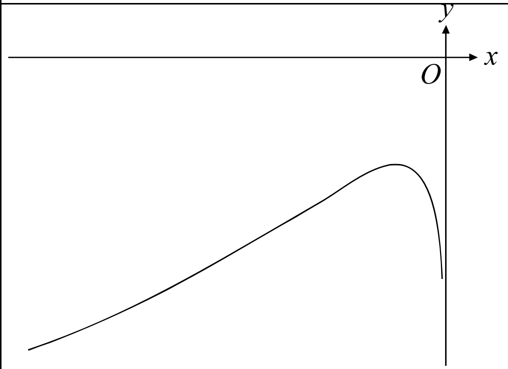
(iii) It is given that C_2 intersects the line $y = \frac{1}{2}x + 1$ exactly once. Find the coordinates of the point of intersection. **[4]**

Question			Answer	Marks	Guidance	
1			$\text{sech}\,2x = \frac{2}{e^{2x}+e^{-2x}}$	B1	For sech2x expression **oe**	
			$u = e^{2x} \Rightarrow du = 2e^{2x}dx$ **or** $x = \frac{1}{2}\ln u \Rightarrow dx = \frac{1}{2u}du$	M1	For differentiating substitution correctly and substituting into *their* integral	
			$\Rightarrow I = \int \text{sech}2x\,dx = \int \frac{2}{e^{2x}+e^{-2x}}dx$ $= \int \frac{2}{\left(e^{2x}+e^{-2x}\right)}.\frac{du}{2e^{2x}}$	A1	For correct integral	
			$= \int \frac{1}{u^2+1}du$			
			$= \tan^{-1} u\ (+c) = \tan^{-1}\left(e^{2x}\right)+c$	M1 A1 **[5]**	For integration to $\tan^{-1}(\ )$ For correct expression (c required)	

Question			Answer	Marks	Guidance	
2	(i)		$r = 0 \Rightarrow \cos\theta = 0, \sin 2\theta = 0$	M1	For $r = 0$ **(soi)** and attempt to solve for θ	
			$\Rightarrow \theta = 0, \frac{1}{2}\pi$	A1	For both values and no others (ignore values outside range)	
				[2]		
2	(ii)		$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\sin\theta\sin 2\theta + 2\cos 2\theta\cos\theta$	M1	For attempt to find $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ using product rule	
			$= 0$	A1	For correct $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ set $= 0$ **soi**	
			Alternatively: $r = 2\cos^2\theta\sin\theta \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\theta} = 2\cos^3\theta - 4\cos\theta\sin^2\theta$			
			$\Rightarrow 2\sin^2\theta\cos\theta = 2(1-2\sin^2\theta)\cos\theta$			
			$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}}\left(\cos\theta = \frac{\sqrt{2}}{\sqrt{3}},\ \tan\theta = \frac{1}{\sqrt{2}}\right)$	A1	For correct value of $\sin\theta$ (**OR** $\cos\theta$ *OR* $\tan\theta$) or decimal equivalent; $\sin\theta = 0.546$ or $\cos\theta = 0.816$ or $\tan\theta = 0.707$	
			$\Rightarrow r = \frac{4}{3\sqrt{3}} = \frac{4}{9}\sqrt{3}$	A1	For correct r or anything that rounds to 0.77	
				[4]		
2	(iii)		$x = r\cos\theta,\ y = r\sin\theta$	M1	For substituting $x = r\cos\theta$ **OR** $y = r\sin\theta$	
			$\Rightarrow r = \frac{x}{r}.2\frac{y}{r}\frac{x}{r}$	M1	For $r^2 = x^2 + y^2$ soi	
			$\Rightarrow \left(x^2+y^2\right)^2 = 2x^2y$	A1	For a correct cartesian equation **Any equivalent form** without fractions	
				[3]		

Question			Answer	Marks	Guidance	
3	(i)		$\tanh 2x \equiv \dfrac{\sinh 2x}{\cosh 2x} \equiv \dfrac{2\sinh x \cosh x}{\cosh^2 x + \sinh^2 x}$	M1	For $\dfrac{\sinh 2x}{\cosh 2x}$ and use double angle formulae	
			$\equiv \dfrac{2\tanh x}{1+\tanh^2 x}$	A1	For division by $\cosh^2 x$ seen	N.B. Tanh($A + B$) not in formula book
				[2]		
3	(ii)		$\dfrac{10t}{\left(t^2+1\right)} = (1+6t)$	M1	For using (i) to obtain equation in t.	
			$\Rightarrow 6t^3 + t^2 - 4t + 1 = 0$	A1	Correct cubic equation	
			$\Rightarrow (t+1)(3t-1)(2t-1) = 0$	M1	Attempt to solve cubic (calculator OK)	
			$\Rightarrow t = (-1), \frac{1}{3}, \frac{1}{2}$	A1	Solution. Ignore any extra values at this stage	
			$x = \frac{1}{2}\ln\dfrac{1+t}{1-t} \Rightarrow x = \frac{1}{2}\ln 2, \frac{1}{2}\ln 3$	M1 A1	For using ln form for $\tanh^{-1}$ Correct 2 values (only) **oe**	
				[6]		
			Alternative: M1		Use exponentials to obtain a quadratic in e^{2x}	
			$e^{4x} - 5e^{2x} + 6 = 0$ A1		Correct	
			$\Rightarrow \left(e^{2x}-2\right)\left(e^{2x}-3\right) = 0$ M1		Solve quadratic	
			$\Rightarrow e^{2x} = 2, \quad 3$ A1		Soln	
			$\Rightarrow 2x = \ln 2, \ \ln 3$ M1		Take logs	
			$\Rightarrow x = \dfrac{1}{2}\ln 2, \quad \dfrac{1}{2}\ln 3$ A1			

Question			Answer	Marks	Guidance	
4	(i)		 $x_2 = 1.3869...$ $x_3 = 1.3938$	B1 B1 B1 **[3]**	For correct value (4 d.p. or better) For correct value. For sketch showing staircase towards α. (Vertical lines do not need to be labelled)	
4	(ii)			B1 B1 **[2]**	For sketch like $y = \frac{1}{2}\left(x^4 - 1\right)$ and $y = x$ (curve or continuation of curve cuts - y axis.) For sketch showing staircase away from α.("Away" means labelling or arrows required.) Labelling means x_1, x_2,... in right place or numeric values.	
4	(iii)		$x_{n+1} = x_n - \dfrac{x_n^4 - 2x_n - 1}{4x_n^3 - 2}$ $1.35 \rightarrow 1.398268$ $\rightarrow 1.395348 \rightarrow 1.395337$ $\Rightarrow 1.3953$	M1 A1 A1 A1 **[4]**	For deriving the iterative formula For correct formula For 1st value For correct 4dp α with 2 iterates equal to 4 dp. (i.e. last two iterates agree to 4dp) **www**	

Question			Answer	Marks	Guidance	
5	(i)		$f'(x)=\dfrac{1}{\sqrt{1+x^2}}+\dfrac{1}{\sqrt{1+\dfrac{1}{x^2}}}\cdot\dfrac{-1}{x^2}$	M1 B1	For attempt to differentiate using chain rule. First term correct	
			$=\dfrac{1}{\sqrt{1+x^2}}\left(1-\dfrac{1}{x}\right)$			
			$=0\Rightarrow x=1$	M1 A1	For attempt to solve their f '(x) = 0 For correct value of x (ignore $x=-1$)**www**	
			$f(1)=2\sinh^{-1}1=2\ln\left(1+\sqrt{2}\right)$	A1 **[5]**	For correct value obtained **www AG**	
5	(ii)			B1	For correct shape in 3rd quadrant only(condone inclusion of the 1st quadrant part given)	
			$\left\{f(x)\geqslant 2\ln\left(1+\sqrt{2}\right),\ f(x)\leqslant -2\ln\left(1+\sqrt{2}\right)\right\}$	B1 B1 **[3]**	For one part of range For other part of range **SC** B1 Both ranges correct but < and > used	

Question			Answer	Marks	Guidance	
6	(i)		$I_n = \left[-x^n \cos x\right]_0^{\pi} + n\int_0^{\pi} x^{n-1}\cos x \, dx$	M1 A1	For attempt to integrate by parts For correct result before limits	
			$= \pi^n + n\left\{\left[x^{n-1}\sin x\right]_0^{\pi} - (n-1)\int_0^{\pi} x^{n-2}\sin x \, dx\right\}$	M1 A1	For attempt at second integration by parts For correct result before limits	
			$\Rightarrow I_n = \pi^n - n(n-1)I_{n-2}$	A1 **[5]**	For correct result **www AG**	
6	(ii)		$I_1 = \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi}\cos x \, dx$	M1	For integrating by parts for I_1	
			$\Rightarrow I_1 = \pi + \left[\sin x\right]_0^{\pi} = \pi$	A1	For correct I_1 SC B1 $I_1 = \pi$ with no working	
			$I_3 = \pi^3 - 6I_1$, $I_5 = \pi^5 - 20I_3$	M1	For substituting n = 3 or 5 in reduction formula	
			$\Rightarrow I_5 = \pi^5 - 20\pi^3 + 120\pi$	A1 **[4]**	For correct result	

Question			Answer	Marks	Guidance	
7	(i)		$a=2,\ b=n$ $c=1,\ \ d=n-1$	B1 B1 B1 **[3]**	for any 2 correct for the third correct for all four correct. Allow values inserted in series. **SC** treat $a=\frac{1}{2}$ etc as MR –1 once	
7	(ii)		$\int_1^n \frac{1}{x}\,dx = \ln n$ $1+\frac{1}{2}+\ldots+\frac{1}{n}<1+\ln n$ $\Rightarrow\ f(n)<1$ (upper bound) $\Rightarrow\ f(n)>\frac{1}{n}$ (lower bound)	B1 M1 A1 A1 **[4]**	For integral evaluated **soi** (Definite integral between 1 and n) For adding 1 *OR* $\frac{1}{n}$ to series For correct upper bound For correct lower bound	
7	(iii)		$f(n+1)-f(n)=\frac{1}{n+1}-\ln(n+1)+\ln n$ $=\frac{1}{n+1}-\ln\left(\frac{n+1}{n}\right)\approx\frac{1}{n+1}-\left(\frac{1}{n}-\frac{1}{2n^2}\right)$ $\approx\frac{1}{n+1}-\frac{2n-1}{2n^2}$ $\approx-\frac{n-1}{2n^2(n+1)}$	B1 M1 M1 A1 A1 **[5]**	For correct expression For combining ln terms For attempt to expand $\ln\left(1+\frac{1}{n}\right)$ Correct expansion of $\ln\left(1+\frac{1}{n}\right)$ For correct expression **AG**	Any expansion of $\ln(1+n)$ oe is 0

Alternative answer to 7(iii)

Question			Answer	Marks	Guidance	
7	(iii)		$\mathrm{f}(n+1)-\mathrm{f}(n)=\dfrac{1}{n+1}-\ln(n+1)+\ln n$	B1	For correct expression	
			$=\dfrac{1}{n+1}-\ln\left(\dfrac{n+1}{n}\right)$			
			$=\dfrac{1}{n+1}+\ln\left(\dfrac{n}{n+1}\right)$	M1	For combining ln terms and attempt to expand	
			$=\dfrac{1}{n+1}+\ln\left(1-\dfrac{1}{(n+1)}\right)$	M1	For attempt to expand $\ln\left(1-\dfrac{1}{(n+1)}\right)$	
			$=\dfrac{1}{n+1}+\left(-\dfrac{1}{(n+1)}-\dfrac{1}{2(n+1)^2}\right)$	A1	Correct expansion of $\ln\left(1-\dfrac{1}{(n+1)}\right)$	
			$=-\dfrac{1}{2(n+1)^2}$		**Max 4**	

Question			Answer	Marks	Guidance	
8	(i)		$q(x) = x + 2$	B1	For correct $q(x)$ **soi oe**	
			$y = \dfrac{A}{x+2} + \frac{1}{2}x + 1$	M1	For expressing y in this form. Allow $cx + d$ for A	
			$\left(-1, \frac{17}{2}\right) \Rightarrow A = 8$	A1	For correct A	
			$y = \dfrac{\frac{1}{2}x^2 + 2x + 10}{x+2} \Rightarrow \; p(x) = \frac{1}{2}x^2 + 2x + 10$	A1 **[4]**	For correct $p(x)$ Allow equal multiples of $p(x)$ and $q(x)$	
			Alternative: $q(x) = x + 2$ B1		For correct $q(x)$ **soi oe**	
			$y = \dfrac{ax^2 + bx + c}{q(x)} = ax + (b - 2a) + \dfrac{c - 2b + 4a}{x+2}$ M1		For division by *their* $q(x)$	1st line of division and 1st term in quotient should be seen for correct method
			$y = \frac{1}{2}x + 1 \Rightarrow a = \frac{1}{2}, b = 2$ A1		For correct a and b **oe**	
			$\left(-1, \frac{17}{2}\right) \Rightarrow c - 2b + 4a = 8 \Rightarrow c = 10$ A1		For correct c **oe**	
8	(ii)		$\frac{1}{2}x^2 + (2 - y)x + 10 - 2y = 0$	M1	For attempt to rearrange as quadratic in x	
			$b^2 - 4ac \geqslant 0 \Rightarrow (2 - y)^2 \geqslant 2(10 - 2y)$	M1	For use of $b^2 - 4ac$ ($\leq$ or $\geq$ or $=$ or $<$ or $>$)	
			$\Rightarrow y^2 \geq 16 \Rightarrow \{y \leq -4, \; y \geq 4\}$	A1 A1	For critical values ± 4 For correct range. (Must be $\leq$ and $\geq$) **www**	
			(pto for alternative)	**[4]**		
8	(iii)		$\left(\frac{1}{2}x + 1\right)^2 = \dfrac{\frac{1}{2}x^2 + 2x + 10}{x+2}$ **OR** $y^2 = \dfrac{4}{y} + y$	B1ft	For a correct equation derived from intersection of C_2 with $y = \frac{1}{2}x + 1$ FT from (i)	
			$\Rightarrow x^3 + 4x^2 + 4x - 32 = 0$ **OR** $y^3 - y^2 - 4 = 0$	M1 A1	For obtaining a cubic Correct cubic	
			$\Rightarrow (2, 2)$	A1 **[4]**	Coordinates correct **www**	

Alternative to 8(ii)

Question			Answer	Marks	Guidance	
8	(ii)		$y=\dfrac{\frac{1}{2}x^2+2x+10}{x+2}$ $\Rightarrow \dfrac{dy}{dx}=\dfrac{(x+2)(x+2)-\left(\frac{1}{2}x^2+2x+10\right)}{(x+2)^2}$	M1	Diffn using quotient rule	
			$=0$ when $(x+2)(x+2)=\left(\frac{1}{2}x^2+2x+10\right)$ $\Rightarrow \frac{1}{2}x^2+2x-6=0 \Rightarrow x^2+4x-12=0$	M1	Attempt to find soln using $\dfrac{dy}{dx}=0$	
			$\Rightarrow (x+6)(x-2)=0$ $\Rightarrow x=2,\ y=4 \quad x=-6,\ y=-4$	A1		
			$\{y\le -4,\ y\ge 4\}$	A1	For correct range. (Must be $\le$ and $\ge$) **www**	
			Alternatively: $y=\frac{1}{2}x+1+\dfrac{8}{x+2}$ $\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{2}-\dfrac{8}{(x+2)^2}$ M1		Diffn using chain rule	
			$=0$ when $\dfrac{1}{2}-\dfrac{8}{(x+2)^2}\Rightarrow (x+2)^2=16$ M1 $\Rightarrow x+2=\pm 4 \Rightarrow x=2$ or -6		Attempt to find soln using $\dfrac{dy}{dx}=0$	
			$\Rightarrow y=4$ or -4 A1 $\{y\le -4,\ y\ge 4\}$ A1		For correct range. (Must be $\le$ and $\ge$) **www**	